On the Product of Two Quadro-Quadric Space-Transformations.

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Although the analogy of the plane theory calls attention to the composition and decomposition of space-transformations, yet the subject has received very slight attention except from Herr S. Kantor,* who has dealt fully with the general case of composition of quadro-quadric transformations. As so often in this theory, the more specialized cases are in some ways more interesting, and worth considering in detail.

The object of this paper is to show that the transformation compounded of two quadro-quadric transformations belongs to one of the following types:

(1) A quarto-quartic, whose fundamental system in either space consists of a double and a simple point, a nodal conic and a simple quartic.

(2) A cubo-quartic, with, in the first space, a double and a simple point, a simple cubic and conic.

(3) A cubo-cubic, with a double point, a simple quartic and conic.

(4) A cubo-cubic, with a simple cubic, conic and straight line.

(5) The general cubo-cubic with a nodal line.

(6) The general quadro-quartic.

(7) The general quadro-cubic.

(8) The general quadro-quadric.

(9) The general linear transformation, including identity.

In the quadro-quadric transformation, the homaloidal family in one space, corresponding to planes in the other, consists of quadrics passing through a fixed conic c and a fixed point p. If the plane of c is taken as $w = 0$, and the coordinates of p are $(0, 0, 0, 1)$, the equations of transformation are those given below (A). There are two distinct ways of specializing this transformation. In the first place, c may break up into a line-pair $a\,b$; equations (A) still apply, and the quadratic function f which occurs in them falls into linear

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factors. In the second place, the fundamental point p may lie on c; then equations (A) cease to be applicable. In this case the quadrics all touch a fixed plane, taken as $z = 0$, different from that of c, at the fixed point p of c. If also c breaks up, the quadrics touch the plane $w = 0$ at the point of intersection of ab, so that p can not lie at this point. In all cases the fundamental systems in the two spaces are of the same nature.

I. Consider then the transformation between the first space and an intermediate space, given by the equations

$$\left. \begin{aligned} x' \;:\; y' \;:\; z' \;:\; w' &= x\,w \;:\; y\,w \;:\; z\,w \;:\; f_2\,(x,\,y,\,z), \\ x \;:\; y \;:\; z \;:\; w &= x'\,w' \;:\; y'\,w' \;:\; z'\,w' \;:\; f_2\,(x',\,y',\,z'), \end{aligned} \right\} \;(A)$$

with fundamental systems

$$\begin{aligned} (c)\;\; 0 &= w \;= f\,, & (p)\;\; 0 &= x \;= y \;= z\,; \\ (c')\;\; 0 &= w' = f'\,, & (p')\;\; 0 &= x' = y' = z'\,;. \end{aligned}$$

and a second transformation between the intermediate and final spaces, given by

$$: Y : Z : W = X'\,W' : Y'\,W' : Z'\,W' : F_2\,(X',\,Y',\,Z'),\ \text{etc.,}$$

with fundamental systems $C',\,P'$; $C,\,P$; where $x'\,y'\,z'\,w'$, $X'\,Y'\,Z'\,W'$ refer to different frames of reference, both in the intermediate space, and are connected by either of the equivalent sets of relations

$$\begin{aligned} X' &= a_1 x' + b_1 y' + c_1 z' + d_1 w', & x' &= A_1 X' + B_1 Y' + C_1 Z' + D_1 W', \\ Y' &= a_2 x' + b_2 y' + c_2 z' + d_2 w', & y' &= A_2 X' + B_2 Y' + C_2 Z' + D_2 W', \\ Z' &= a_3 x' + b_3 y' + c_3 z' + d_3 w', & z' &= A_3 X' + B_3 Y' + C_3 Z' + D_3 W', \\ W' &= a_4 x' + b_4 y' + c_4 z' + d_4 w', & w' &= A_4 X' + B_4 Y' + C_4 Z' + D_4 W'. \end{aligned}$$

Compounding these, we have the quarto-quartic transformation No. 1, between the first and final spaces, given by

$$\begin{aligned} X : Y &: Z : W \\ &= \{(a_1 x + b_1 y + c_1 z)\,w + d_1 f\}\,\{(a_4 x + b_4 y + c_4 z)\,w + d_4 f\} \\ &:\{(a_2 x + b_2 y + c_2 z)\,w + d_2 f\}\,\{(a_4 x + b_4 y + c_4 z)\,w + d_4 f\} \\ &:\{(a_3 x + b_3 y + c_3 z)\,w + d_3 f\}\,\{(a_4 x + b_4 y + c_4 z)\,w + d_4 f\} \\ &:F_2[\{(a_1 x + b_1 y + c_1 z)\,w + d_1 f\},\ \{(a_2 x + b_2 y + c_2 z)\,w + d_2 f\}, \\ &\qquad\qquad\qquad \{(a_3 x + b_3 y + c_3 z)\,w + d_3 f\}], \end{aligned}$$

with a fundamental system h in the first space consisting of: a double point p; a simple point q corresponding to P', given by

$$\frac{1}{d_1}\,(a_1 x + b_1 y + c_1 z) = \frac{1}{d_2}\,(a_2 x + b_2 y + c_2 z) = \frac{1}{d_3}\,(a_3 x + b_3 y + c_3 z) = -\frac{f}{w}\,;$$

a nodal conic c; and a simple quartic c_4 corresponding to C', having two branches through p and meeting c four times.

The results for the reverse transformation are similar.

Either or both of the component transformations can be specialized in the ways mentioned above; any combination of these gives a particular case of No. 1, with a corresponding specialization of h.

Specialization also occurs when the two fundamental systems in the intermediate space are specially related. If c', C' intersect in k points ($k = 1, 2$ or 4), then c_4 breaks up into k generators of the cone (p, c) and a proper curve c_{4-k}. Any other relation of incidence gives a different transformation of lower degree, and not a particular case of No. 1. These transformations of lower degree occur when the quartic homaloidal family (s_4) in the first space breaks up into a fixed part and a family of lower degree. The fixed part corresponds to part of the system c', p', which must lie on all the quadrics of the second family in the intermediate space (corresponding to planes in the final space), and must therefore coincide with part of the system C', P'. Besides (I, above) the arrangements which give no reduction of degree, which have been sufficiently discussed, we have to consider the following cases and their combinations:

II. Component transformations not specialized.
  (i) p' lies on C'.
 (ii) $p' \equiv P'$.
  (iii) $c' \equiv C'$.

III. Components specialized; c', C' break up with a common line $b' \equiv B'$.
  (i)   No further specialization.
  (ii) p' lies on A'.
  (iii) $p' \equiv P'$.

IV. Components specialized; p' lies on c'.

II. *Components not specialized.* (i) *p' lies on C'.* In the first space, s_4 breaks up into the plane of c and a cubic surface with a node at p and containing c, q; instead of c_4 we have a plane cubic curve with a node at p. In the final space, S_4 does not break up, but instead of Q (representing p') it contains a generator of the cone (P, C). We have the cubo-quartic transformation No. 2.

If also P' lies on c', in the first space q is replaced by a generator of the cone (p, c); the transformation is cubo-cubic No. 4. The node of s_3 at p is now a necessary consequence of the passage of the surface through this generator and the cubic curve. This is a particular case of the bilinear transformation in which the fundamental sextic breaks up as in the diagram, which shows the actual intersections of the system h.
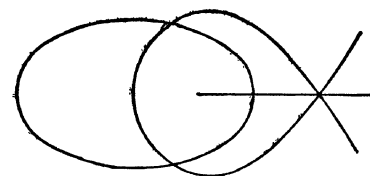
This includes the reciprocal transformation in which the sextic breaks up into the six edges of a tetrahedron. Analytically, the transformation

$$X : Y : Z : W = \frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{w}$$

can be compounded of

$$x' : y' : z' : w' = xw : yw : zw : yz \text{ and } X : Y : Z : W = y'z' : z'x' : x'y' : x'w'.$$

Here, in the intermediate space, both conics break up, and each fundamental point lies at the point of intersection of the other line-pair.

(ii) $p' \equiv P'$. In the first space, s_4 breaks up into the plane of c and a cubic surface with a node at p and containing c, c_4; the simple point q disappears. The reverse is similar, and we have the cubo-cubic transformation No. 3.

(iii) $c' \equiv C'$. Here s_4 breaks up into the cone (p, c) and a quadric containing c, q; we have a general quadro-quadric transformation No. 8. If also $p' \equiv P'$, then s_4 breaks up into the cone (p, c), the plane of c and a variable plane; and we have the general linear transformation No. 9.

III. $c' \equiv a' + b'$, $C' \equiv A' + b'$. The cone (p, c) breaks up into two planes $p\,a$, $p\,b$ corresponding to a', b'; then s_4 breaks up into the plane $p\,b$ and a cubic surface s_3 with a as a nodal line and containing b, p, q.

(i) *No further specialization.* To A' corresponds a pair of straight lines, viz., to the point $A'\,b'$, a generator lying on the plane $p\,b$ and not on s_3, and to the rest of A', a straight line lying on s_3, meeting a and this generator. We have the general cubo-cubic transformation No. 5, with a fundamental nodal line, two simple lines meeting it and two points.

If also the point $a'\,b'$ coincides with the point $A'\,b'$, the two simple lines meet on the nodal line; if the line-pairs are coplanar, one of the points lies on one of the simple lines; i. e., at this point s_3 touches a fixed plane.

(ii) p' *lies on* A'. Then s_3 breaks up into the plane of c and a quadric s_2 containing a, p, q. Since A' is a generator of the cone (p', c'), there corresponds to it a single point of b through which s_2 passes. In the final space, Q is replaced by a generator, so that P is not an isolated point. We have the general quadro-cubic transformation No. 7.

If also P' lies on a', the transformation is quadro-quadric.

(iii) $p' \equiv P'$. Here s_3 breaks up into the plane of c and a quadric; the transformation is quadro-quadric.

IV. p' *lies on* c'. Then the cone (p', c') breaks up into the plane $w' = 0$ of the conic and another plane $z' = 0$ which touches the first quadric family in

the intermediate space at p'. A surface containing c' has the degree of its corresponding surface diminished by two, as if it did not contain p', unless it touches z', when the degree is diminished by three.

The only new case occurs when the quadrics of the second family in the intermediate space touch z' at p'. This can happen in two ways:

(i) P' lies on C' and coincides with p', the planes Z', z' also coinciding; this leads to a quadro-quadric transformation of the same type as the components.

(ii) $C' \equiv A' + B'$, and the line-pair lie in the plane z' and intersect at p'. Then, in the first space, s_4 breaks up into the plane of c counted twice and a quadric touching z at p and containing q and two points corresponding to A', B'. In the final space, S_4 has P as a double point and A, B as nodal lines; to c' there correspond a double line joining P to the point (A, B) and a conic through P meeting A, B. We have the general quadro-quartic transformation No. 6.

Analytically, the transformation
$$X : Y : Z : W = yz \quad : zx \quad : xy \quad : w\,(px + qy + rz),$$
$$x : y : z : w = YZF : ZXF : XYF : XYZW, \qquad F \equiv pYZ + qZX + rXY,$$
can be compounded of
$$x \ : y \ : z \ : w \ = x'p' : y'p' : z'p' : x'w', \qquad p' \equiv px' + qy' + rz',$$
$$x' : y' : z' : w' = YZ : ZX : XY : XW.$$

As an example of the possible extension of the theory, consider all the cubic transformations with a fundamental nodal line a. There are four such: a cubo-quintic with six fundamental points besides the nodal line, a cubo-quartic with a generator and four points, a cubo-cubic with two generators and two points, and a cubo-quadric with three generators. To the cubic homaloidal family of the first of these apply a quadro-quadric transformation whose fundamental system consists of p (one of the six points), a and another line b meeting a and passing through q (another of the six points). The family is transformed into another cubic homaloidal family with b' as a nodal line, and, corresponding to q, a simple line meeting b' and absorbing the simple point p', and four other simple points. This is the homaloidal family of the second of the transformations. If we now repeat the process, we again lose two points and gain a line, and the family becomes the homaloidal family of the cubo-cubic transformation. Another repetition leads to the last homaloidal family with three generators, and as there are no fundamental points left, we can not continue the process; but at this or any stage after the first, we can use the nodal

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line, a generator and any other common point as the fundamental system of a quadro-quadric transformation which transforms the family into a family of quadrics.

Since all quadric transformations are compounded of quadro-quadric ones, it follows that if the homaloidal family of a given transformation can be converted into a quadric family by a series of quadric transformations, then the given transformation is compounded of quadro-quadric transformations only.

With reference to the list of cubic transformations given in a previous paper,* and the statements there made as to their composition, it is clear that a cubic family can be converted into a quadric family if it belongs to one of the following species; viz., the fundamental system contains either:

(i)   a nodal line (Nos. 1, 5, 11, 27 of that list), or

(ii)   a conic and a node not lying on it (Nos. 21, 30, 36, 38, 57), or

(iii)   a conic touching the fixed plane at a binode (Nos. 16, 20, 22, 33, 37, 45, 49, 58).

In Nos. 46, 61, using two rays and a point of contact as the fundamental system of a quadro-quadric transformation (and in No. 55, two rays and a simple point), we convert the cubic families into families of species (iii). A similar treatment of No. 48 leads to species (ii). In No. 54, using p, C_1 and a line through q meeting C_1, we obtain species (i).

In No. 43, apply the cubo-quadric transformation in which the three generators are C_1 and the two rays, and we obtain species (i); No. 31 can be treated in the same way, if two of the generators coincide in the ray of contact. The same treatment applied to No. 41 leads to No. 50; for the latter, apply a quadro-quadric transformation using pq, pr and the point of contact, and we obtain species (ii).

In the same way, if the fundamental system contains a conic and any other point, or a node and a straight line not passing through it, the family can be transformed into another cubic family belonging to the same or a different transformation.

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